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POLYOMINOES AND ANIMALS: SOME RECENT RESULTS*

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Abstract

We give a survey of recent works relating algebraic languages and formal power series with the enumeration of polyominoes (and animals). More precisely, encoding these structures with words yields new exact results.

1. Introduction

Let Ω be a class of combinatorial objects. Let us suppose that they are enumerated by the integer a_n according to the value n of some parameter p. Let us further suppose that the corresponding generating function $f(t) = \sum_{n \geq 0} a_n t^n$ is *algebraic*.

Schützenberger's methodology in [48,49], consisting in first constructing a bijection between the objects Ω and the words of an *algebraic language*, accounts for the explanation for the algebraic nature of the generating function. Let ω be an object in Ω. Then the parameter p of ω turns out to be a number of letters in the corresponding word coding of ω. This methodology was first illustrated by Cori [16], then by Cori and Vauquelin [17] about Tutte formulas on planar maps. The reader will find an introduction to the topic in [10,30] and a synthesis by Viennot in [52]. Recently, this method has been effectively used to code and count *polyominoes*, which can be described as a finite connected union of *cells* (unit squares) in the plane $\mathbb{N} \times \mathbb{N}$; see [31] for instance. A polyomino is displayed in fig. 1.

The most often studied parameters are the *perimeter*, which is the length of the border of the polyomino, and the *area*, which is the number of cells.

Counting polyominoes is a problem in combinatorics which more often than not remains unsolved. Yet, some exact formulas dependent on one parameter only (e.g. *either* the perimeter *or* the area) are proved for some particular types of polyominoes. The reader is referred to [37,38] for examples. However, all the research on polyominoes so far has led one to believe that it is a more difficult problem when it comes to solving the distribution for two parameters at the same time (e.g. *both* the perimeter *and* the area).

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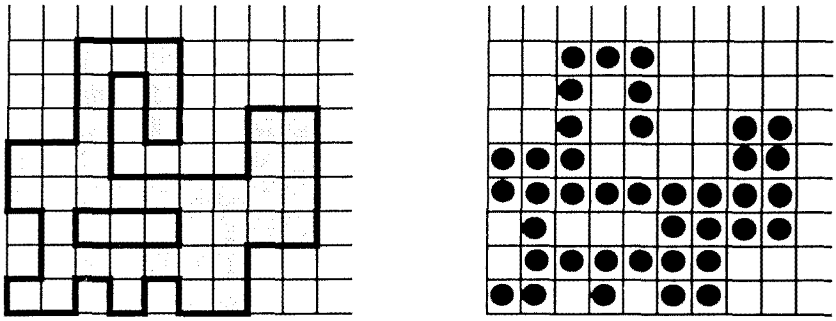


Fig. 1. A polyomino and an associated animal.

This problem is also well known in statistical physics. Usually, physicists consider animals instead of polyominoes, an equivalent object obtained by taking the center of each elementary cell (see fig. 1).

Below, we give several examples in which Schützenberger's methodology has solved open problems in the field of polyominoes.

We shall begin with a brief review of the problems around polyominoes and also introduce the methodology. We shall conclude with new features.

2. Polyominoes and animals

The study of polyominoes contains a large set of problems. Their study is connected to partition problems, but the first book on this subject is due to Golomb [31] in 1965. It was preceded by some papers by Gardner in 1958 in the Scientific American [29]. See also the nice paper by Klarner, "My life among polyominoes" [37]. There are two classes of problems when dealing with polyominoes. The first one aims at enumerating them according to the perimeter and/or the area, and the second at spanning the plane with a set of polyominoes having a given area [32,53].

This does not lead to enumeration problems but rather to algorithms allowing us to obtain a polyomino by spanning it with a smaller one [5,6], by superimposing rectangles [53]. Here are some possible applications:

- design of VLSI [14], the shadow of a VLSI circuit is a polyomino,
- storage of images [1,13], the periphery is a polymino.

We are interested in enumerating polyominoes. Generally speaking, only asymptotic results are known, the latest ones being Guttmann's [34]. Thus, many people take particular polyominoes into account in order to get some approaches to the general problem. To describe particular cases, let us define a *column* (respectively, *row*) of a polyomino as the intersection with an infinite vertical (respectively, horizontal) strip of cells. A polyomino is *column-convex* (respectively, *row-convex*) if every colum (respectively, row) is connected. It is *convex* if it is both row- and column-convex. See the examples in fig. 2.

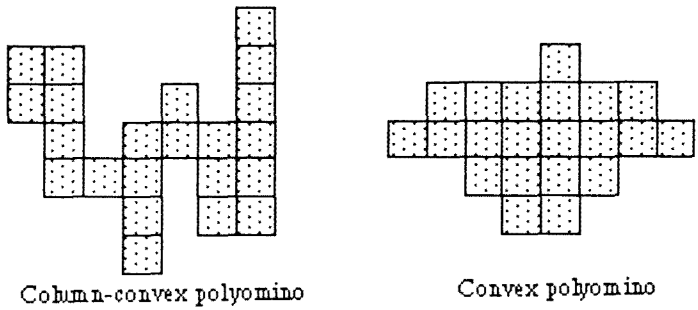


Fig. 2. Some types of polyominoes.

An *animal* is a set of points of $\mathbb{N} \times \mathbb{N}$ such that every pair of points of the animal can be connected by a path (sequence of points) included in the animal and having elementary steps North, East, South and West. Animals are related to the percolation problem, and many results have been published on this subject [45]. Physicists attempt to find some relations for the number a_n of animals having area or perimeter n. They look for asymptotic results in the form $a_n \approx \mu^n n^{-\theta}$. The exponent θ is called the *universality class* of the model and n the connecting constant.

Recently, the interest was in *directed* animals. They are related to some gas lattice models. An animal is said to be *directed* if it contains a set of s points (called roots or source points) lying on the line $x + y = s - 1$, such that any other point in the animal can be reached from one of the roots by a path making only North or

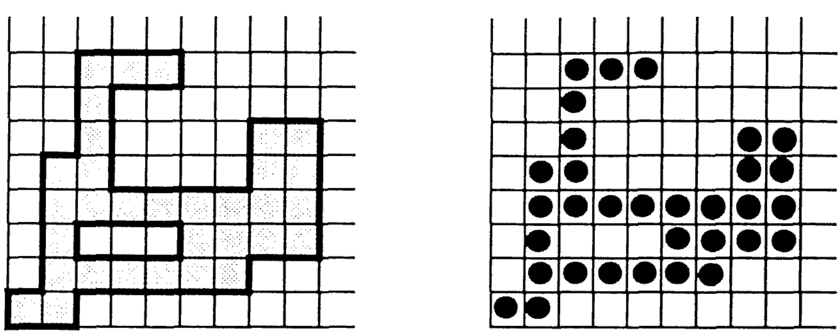


Fig. 3. A directed polyomino and an associated animal.

East steps in the lattice plane within the animal (see fig. 3). The surprising result was that exact results can be found for this class of animals [41,25,26,36]. See [51] for a survey.

Note that polyominoes are obtained from animals by placing a unit square with vertices at integer points for each point of the animal. Thus, we shall say that a polyomino is directed if the associated animal is directed, and in the following we shall only use the word polyomino. Let P be a polyomino. The enumeration is made according to the following parameters:

- the *bond* perimeter $p(P)$, that is, the length of the border of the polyomino,
- the *site* perimeter $s(P)$, that is, the number of squares (respectively, unit cells) outside and adjacent to the boundary of the polyomino (respectively, animal),
- the *area* $a(P)$, that is, the number of squares (respectively, unit cells) of the polyomino (respectively, animal).

3. Schützenberger's methodology

Let $X = \{x_1, x_2, \ldots, x_k\}$ be an alphabet. We denote by X^* the free monoid generated by X, that is, the set of words (finite sequences of letters from X). The *empty word* is denoted by ε. The number of occurrences of the letter x in the word w is denoted by $|w|_x$, the length (number of letters) of w by $|w|$. Let Ω be a class of objects for which a parameter π is to be studied. Schützenberger's methodology is based upon four steps:

(1) code the objects of Ω by the words of an algebraic language $\mathcal{L}$ preserving π;

(2) write out a non-ambiguous grammar $\mathcal{G}$ generating the language $\mathcal{L}$;

(3) solve the algebraic system associated to $\mathcal{G}$ in commutative variables, obtaining a generating function $\mathcal{F}$ (or a functional equation) for the language $\mathcal{L}$;

(4) compute, using $\mathcal{F}$, an exact formula or an asymptotic expression for the number of objects in Ω having a given value for the studied parameter π.

For example, let Ω be the class of stack polyominoes. A *stack polyomino* S is a convex polyomino given by two paths η and λ from $(0, 0)$ to $(k, 0)$. The path η makes only East steps. In the first part, λ makes only North and East steps, then after an East step makes only South and East steps (see fig. 4).

The first step consists of coding the stack polyominoes using a word w of $\{x, a\}^*$, such that

(i) w is in $(x + a)^*$,

(ii) $|W|_a$ is even.

These words constitute the language $\mathcal{L}$. This coding is immediately obtained translating the path λ: each East step is translated by the letter x with the exception of the middle one, and each North or South step by a letter a, with the exception of the first and last (see fig. 6). Then, we have $p(S) = |w|_a + 2\,|w|_x + 4$.

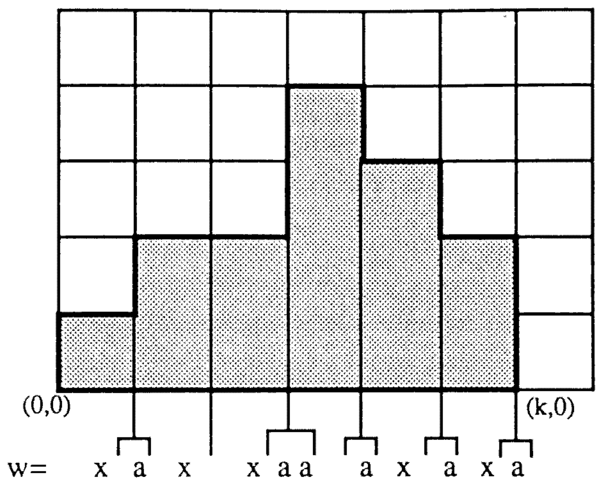


Fig. 4. A stack polyomino and its coding.

In the second step, we write the non-commutative system of equations associated with the previous language

$$L = a\,L\,a\,L_1 + x\,L + \varepsilon, \tag{1}$$

$$L_1 = \varepsilon + x\,L_1, \tag{2}$$

where

$$L = \sum_{w \in \mathcal{L}} w.$$

The first equation means that a non-empty word w in $\mathcal{L}$ has the form $w = xw'$, with w' in $\mathcal{L}$ or $w = aw_1aw_2$, with w_1 in $\mathcal{L}$ and w_2 in $\{x\}^*$.

In the third step, by commuting the variables, we obtain the commutative image of $\mathcal{L}$

$$l(x, a) = \frac{1-x}{(1-x)^2 - a^2}.$$

This function enumerates the stack polyominoes according to its height and width. From it, one can easily prove as an example of the fourth step, the following

PROPOSITION 1

The number of stack polyominoes whose perimeter is $2p + 4$ is the Fibonacci number F_{2p}.

4. Enumeration of polyominoes using this methodology

Knuth asked the question: what is the number of convex polyominoes [38]? In 1984, Delest and Viennot enumerated these according to the perimeter [24]. They showed that the number of convex polyominoes whose perimeter is $2n+8$ is given by

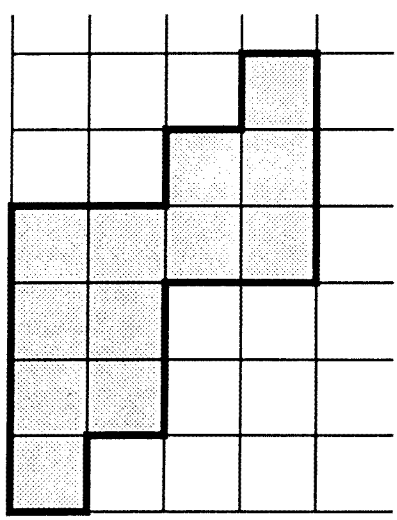
$$p_4 = 1, \quad p_6 = 2,$$

$$p_{2n+8} = (2n+11)4^n - 4(2n+1)\binom{2n}{n}, \qquad \text{for } n \geq 0.$$

This result was recently found again by Enting and Guttmann [35], and Lin and Chang [40]. On the other hand, according to the area, there is only an asymptotic result [34]

$$g_p = 2.67564 \cdot (2.30914)^n.$$

Following this work, since 1984, we investigated several kinds of polyominoes which are related to some properties of convexity. Firstly, we examined the *parallelogram polyominoes*, which are defined by two non-intersecting paths beginning and ending at the same points and making only North and East steps (see fig. 5).



Perimeter 20
Site perimeter 16
Area 12

Fig. 5. A parallelogram polyomino.

The number of such polyominoes with perimeter $2n+2$ is known to be the Catalan number C_n. We proved [23] that the number of such polyominoes having perimeter $2n$ and site perimeter $2n-k$ is

$$C_{n,k} = \frac{2}{k+2}\binom{n-2}{k}\binom{n}{k+1}.$$

For column-convex polyominoes, it was well known [36b] that the generating function according to the area was rational. In [18], the generating function according to the bond perimeter is proved to be algebraic. Its expression needs a full page of formulas.

For directed animals, the first study was made by Dhar et al. [27]. Exact results were proved successively by Dhar [26], Hakim and Nadal [36], and lastly, using combinatorics, by Gouyou-Beauchamps and Viennot [33], and very recently by Betrema and Penaud [11]. Finally, the following results are known:

- the number of directed animals having area n is

$$a_n = \sum_{i=0}^{n-1}\binom{n-1}{i}\binom{i}{\lfloor i/2 \rfloor};$$

- the number of directed animals having area $n+1$ with compact source is 3^n.

However, no exact result concerning the perimeter is known.

In the case of directed column-convex animals [19], one can find exact results for the three parameters. The most surprising result was that the number of those having an area n is the Fibonacci number of rank $2(n-1)$. For fully diagonal compact animals [19] (i.e. directed and with diagonal compact (see fig. 6)), Privman and Svrakič [47] gave the generating function according to the area. In [20], we gave it according to the two perimeters.

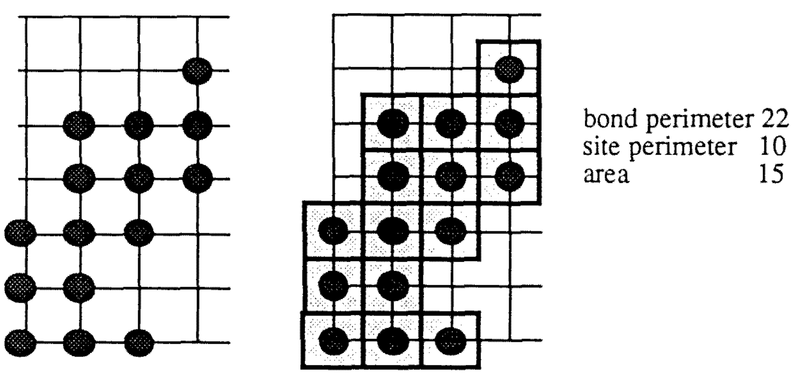


Fig. 6. A fully diagonal compact animal and the associated polyomino.

Another most surprising result was that the number of such polyominoes having one root and a site perimeter equal to $n+1$ is

$$d_n = \frac{1}{2n+1}\binom{3n}{n}.$$

This number is the number of ternary trees having n internal nodes. This result has also been recently proved by Penaud [44].

In fact, in many cases exact results according to the perimeter are well known, and according to the area there is only asymptotic or no result. In other cases, the situation is just the opposite. For example, the generating function for column-convex polyominoes according to the area is rational, but the one according to the perimeter is algebraic. This has set us wondering. We have noted that the bijection between polyominoes and algebraic languages preserves the parameter area when the coding is made according to the perimeter. Thus, since 1987 we have searched for some methods relating area and perimeter in polyominoes enumeration. In the last section, we will show an extension of the Schützenberger methodology which allows us to deduce the generating function according to the two parameters together. However, first we give one more simple example.

5. Another example: The parallelogram polyominoes

In this section, we explain how to obtain a coding for parallelogram polyominoes preserving the four parameters [24,18]. A *path* is a sequence of points in $\mathbb{N} \times \mathbb{N}$. A *step* of a path is a pair of two consecutive points in the path. A *Dyck path* is a path $w = (s_0, s_1, \ldots, s_{2n})$ such that $s_0 = (0,0)$, $s_{2n} = (2n, 0)$, having only steps North–East ($s_i = (x, y)$, $s_{i+1} = (x+1, y+1)$) or South–East ($s_i = (x, y)$, $s_{i+1} = (x+1, y-1)$). A *peak* (respectively, *trough*) is a point s_i such that the step (s_{i-1}, s_i) is North–East (respectively, South–East) and the step (s_i, s_{i+1}) is South–East (respectively, North–East). The *height* $h(s_i)$ of a point s_i is its ordinate.

A *Dyck word* is a word $w \in \{x, \bar{x}\}^*$ satisfying the following two conditions:

(i) $|w|_x = |w|_{\bar{x}}$;

(ii) for every factorization $w = uv$, $|u|_x \geq |u|_{\bar{x}}$.

Classically, a Dyck path having length $2n$ is coded by a Dyck word of length $2n$, $w = x_1 \ldots x_{2n}$: each North–East (respectively, South–East) step (s_{i-1}, s_i) corresponds to the letter $x_i = x$ (respectively, $x_i = \bar{x}$). The peaks (respectively, troughs) of a Dyck path correspond to the factors $x\bar{x}$ (respectively, $\bar{x}x$) of the associated Dyck word. The Dyck path shown in fig. 7 is coded by the Dyck word

$$w = xxxx\bar{x}\bar{x}xx\bar{x}\bar{x}\bar{x}xxx\bar{x}xxx\bar{x}\bar{x}\bar{x}.$$

A parallelogram polyomino P can be defined by the two sequences of integers $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_{n-1})$, where a_i is the number of cells belonging to the ith column and $(b_i + 1)$ is the number of cells adjacent to columns i and $i+1$.

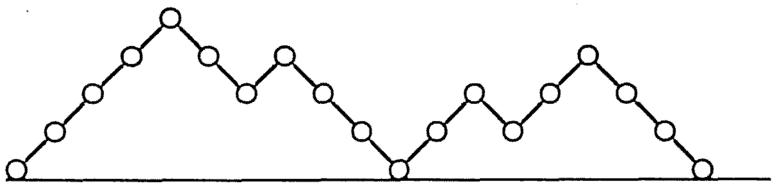


Fig. 7. A Dyck path.

The Dyck word $\mu(P)$ is the Dyck word associated to the Dyck path having n peaks, whose heights (respectively, troughs) are $a_1, \ldots, a_n$ (respectively, $b_1, \ldots, b_{n-1}$). Note that μ associates the parallelogram polyomino of fig. 5 to the Dyck path of fig. 6. It is very easy to prove that μ is a bijection preserving the four parameters:

- if $p(P) = 2n + 2$, then $|\mu(P)| = 2n$;
- if $s(P) = k$, then $|\mu(P)| - |\mu(P)|_{\bar{x}\bar{x}x} - |\mu(P)|_{\bar{x}xx} = k$;
- if $a(P) = r$, then the sum of the heights of the peaks in $\mu(P)$ is r;
- if the width of P is h, then $\mu(P)$ has h factors $x\bar{x}$.

In the second step, we write the non-commutative equation associated to the language

$$D = x\bar{x} + xD\bar{x} + x\bar{x}D + xD\bar{x}D.$$

From this equation, taking the commutative image, it is easy to prove that the number of such polyominoes having a bond perimeter $2n + 2$ is the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Let us now explain how to obtain the generating function according to the bond perimeter and the width. First marking with a letter, say t, every factor $x\bar{x}$, gives

$$D = xt\bar{x} + xD\bar{x} + xt\bar{x}D + xD\bar{x}D. \tag{3}$$

From this point, take the commutative image and apply the morphism $\eta(x) = \eta(\bar{x}) = x$, $\eta(t) = t$. Then we obtain the equation in commutative variables

$$d(x, t) = x^2t + x^2d(x, t) + x^2td(x, t) + x^2d^2(x, t),$$

in which

$$d(x, t) = \sum_{h \geq 0} \sum_{n \geq 0} d_{n,h} x^{2n} t^h$$

and $d_{n,h}$ is the number of parallelogram polyominoes having width h and perimeter $2n+2$. From this, it is easy to deduce that

$$d_{n,h} = \frac{1}{n}\binom{n}{h}\binom{n}{h-1}.$$

In the last section, we will show a transformation which permits us to take into account the area using the last equation (3).

6. q-series and compiling

Let P be a polyomino and let us suppose that it is coded by a word w such that $|w|$ is the perimeter of P. Let $Q(w)$ be $q^{a(P)}$, where $a(P)$ is the area of P. Let us consider the formal power series

$$\sum_{w \in L} Q(w)w.$$

Taking the commutative image, we obtain an enumerating function which turns out to be a series in two variables

$$f(x;q) = \sum_{n \geq 0} \sum_{p \geq 0} f_{n,p}\, x^n q^p,$$

in which $f_{n,p}$ is the number of polyominoes whose perimeter is $2n$ and area is p. Note that such a generating function is related to q-series in combinatorics. There is a vast literature on q-calculus and q-series. A nice introduction to the subject can be found in the paper by Foata [28]. Here, we give just a few features.

The q-analogue of an integer n is the polynomial

$$[n] = 1 + q + q^2 + \ldots + q^{n-1},$$

and the q-analogue of an n factorial is

$$[n]! = \prod_{i=1}^{n} [i].$$

In some way, a q-series is a series s in $\mathbb{C}[[X, q]]$,

$$s(x;q) = \sum_{n \geq 0} \alpha_n(q) x^n,$$

where $\alpha_n(q)$ is some function in $\mathbb{C}[[q]]$ in which the classical q-analogue $[n]$ comes up. The recent book by Andrews [2] introduces some applications of q-calculus to number theory and physics. A very fruitful way of obtaining some combinatorial interpretation of q-analogues of classical numbers is by replacing the ordinary counting of the corresponding objects by q-counting. If C is a set of objects, the cardinality of C is

$$|C| = \sum_{x \in C} 1.$$

A q-counting of the elements of C will be the formal power series

$$|C|_q = \sum_{x \in C} q^{s(x)},$$

where s is a statistics on the elements of C.

What we need now is to have a means of relating grammars to q-series. In other words, knowing the word coding the polyomino we must construct its translation, which is a word "shuffled" with letter q.

In computer science, the compiler theory (more precisely the attribute grammars which were introduced by Knuth [39]) permits us to associate a translation to a word of an algebraic language. The interest of the method is that every translation is defined locally on every rule (every monomial) of the grammar (equations). Thus, the problem of finding recurrences on a polyomino according to the area is transformed into a very local problem on some particular configurations of the polyomino.

7. q-grammars and enumeration

In [22], we define what we call a q-grammar. For short, just consider that we associate to every monomial of a non-commutative equation a translation function τ called *attribute*. Then the pair (S, τ), where S is the non-commutative system of equations, is called a q-grammar. The q-analogue of the enumerating function L (denoted by ${}^qL$) is the series in $\mathbb{B} \ll X \cup \{q\} \gg$ defined by

$${}^qL = \sum_{w \in L} \tau(w).$$

The attribute τ is such that if we substitute to each q the value 1, then we merely obtain the word w. In many cases, $\tau(w)$ will appear as a shuffle of the word w and a word of $\{q\}^*$. Similarly, the function ${}^1L$ is merely the enumerating function of L.

The commutative image of the series ${}^qL$ is the series over $X \cup \{q\}$ defined by

$${}^ql(X) = \sum_{i_1 \geq 0, \ldots, i_k \geq 0} \lambda_{i_1, \ldots, i_k}(q) x_1^{i_1} \ldots x_k^{i_k}.$$

The coefficient $\lambda_{i_1, \ldots, i_k}(q)$ is in $\mathbb{C}[\{q\}]$ and often rational in q in our examples. The series ${}^ql(X)$ is clearly a q-series. Therefore, it ends up as being a natural way of relating a q-series to an algebraic ordinary generating function. Now we give two very simple examples. First, in the case of stack polyominoes, we write the associated attribute to each monomial of the system of eqs. (1) and (2):

$$\tau(L) = q^{|\tau(L)|_x} a\,\tau(L) a\,\tau(L_1) \qquad \text{(associated to } L \to aLaL_1\text{)},$$

$$\tau(L) = qx\,\tau(L) \qquad \text{(associated to } L \to xL\text{)},$$

$$\tau(L) = \varepsilon \qquad \text{(associated to } L \to \varepsilon\text{)},$$

$$\tau(L_1) = qx\,\tau(L_1) \qquad \text{(associated to } L_1 \to xL_1\text{)},$$

$$\tau(L_1) = \varepsilon \qquad \text{(associated to } L_1 \to \varepsilon\text{)}.$$

From [22], it can be easily proved that ${}^q l(x, a)$ is a solution of the system

$${}^q l(x, a) = qx\,{}^q l(x, a) + a^2\,{}^q l(xq, a)\,{}^q l_1(x, a) + \varepsilon,$$

$${}^q l_1(x, a) = qx\,{}^q l_1(x, a) + \varepsilon.$$

By solving this system, we obtain the following

PROPOSITION 2

The number of stack polominoes having perimeter $2p + 2$ and area n is the coefficient of $x^p q^n$ in the q-series

$$S(x;q) = \sum_{k \geq 0} \frac{x^{k+1} q^{k+1}(1 - xq^{k+1})}{\prod_{i=1}^{k+1} (1 - xq^i)^2}$$

Using eq. (3), it is also easy to deduce a q-equation for parallelogram polyominoes. First, we write the associated attribute to each monomial:

$$\tau(D) = qxt\bar{x} \qquad \text{(associated to } D \to xt\bar{x}\text{)},$$

$$\tau(D) = q^{|\tau(L)|_t} x\,\tau(D)\,\bar{x} \qquad \text{(associated to } D \to xD\bar{x}\text{)},$$

$$\tau(D) = qxt\bar{x}\,\tau(D) \qquad \text{(associated to } D \to xt\bar{x}D\text{)},$$

$$\tau(D) = q^{|\tau(D)|_t} x\,\tau(D)\,\bar{x}\,\tau(D) \qquad \text{(associated to } D \to xD\bar{x}D\text{)}.$$

From this, we deduce [21]

Table 1

Exact enumeration of polyominoes

Polyomino	Perimeter	Area
Stacks	Exercise	Euler 1748, Gauss 1863 Sylvester 1884 Temperley 1952, 1956 Wright 1968 Derrida, Nadal 1984
Parallelogram	Pólya 1969 Kreweras 1970 Delest, Gouyou-Beauchamps, Vauquelin 1987 (site and bond)	(particular case of *quasi-partitions*: Auluck 1951, Andrews 1981), Pólya 1969 Gessel 1980 Delest, Fedou 1988 (area and width)
Directed convex	Chang, Lin 1988 (width and length) Bousquet-Mélou 1990	Bousquet-Mélou, Viennot 1990 (area, width and length)
Convex	Delest, Viennot 1984 Kim, Stanton 1988 Enting, Guttmann 1988, 1989 Chang, Lin 1988 Lin 1988 (width and length)	(*asymptotic results*: Klarner, Rivest 1974, Bender 1974) Bousquet-Mélou 1990 (area, width and length)
Column-convex	Delest 1987	Klarner 1965, 1967 Stanley 1978, 1986 Delest 1987 (area and width) Privman, Forgacs 1987 Privman, Svrakič 1989 (area and length)
Directed, column-convex	Delest, Dulucq 1987 (site and bond)	Delest, Dulucq 1987 Barcucci, Pinzani, Rodella 1990
Fully diagonal, compact	Delest, Fédou 1988 (site and bond) Penaud 1990	Bhat, Bhan, Singh 1988 Privman, Svrakič 1988
Directed		Nadal, Derrida, Vannimenus 1982 Hakim, Nadal 1982 Dhar, Phani, Barma 1982 Dhar 1982, 1983 Viennot 1985 Gouyou-Beauchamps, Viennot 1988 (area and width) Betrema, Penaud 1990

THEOREM 11

The number of skew Ferrers diagrams having area n and p columns is the coefficient of $t^p q^n$ in the q-series

$$^{q}s(t) = (1-q)\varphi_0\left(\frac{qt}{(1-q)^2}\right),$$

where $\varphi_0(x)$ is the quotient of two basic Bessel functions

$$\varphi_0(x) = \frac{{}_qI_1(x)}{{}_qI_0(x)},$$

in which the basic Bessel function is defined by

$${}_qI_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n+\nu}{2}} x^{n+\nu}}{[n]![n+\nu]!}.$$

Recently, using this method, Bousquet-Mélou [12] has given a generating function for *convex polyominoes according to the area.*

8. Conclusion

In table 1, we give a list of authors of polyomino enumeration which is due to Delest, Penaud and Viennot and pictured in [42]. A remarkable fact of all these codings with words is that they are very efficient on planar pictures and especially for polyominoes. The interest of these codings is the interplay between computer science, combinatorics and physics. Finally, we note that most of the results were obtained using symbolic calculus (in particular, MAPLE from Waterloo University) and also using the book by Sloane [50].

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